

ROTA-BAXTER HOM-LIE-ADMISSIBLE ALGEBRAS

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ABSTRACT. We study Hom-type analogs of Rota-Baxter and dendriform algebras, called Rota-Baxter G -Hom-associative algebras and Hom-dendriform algebras. Several construction results are proved. Free algebras for these objects are explicitly constructed. Various functors between these categories, as well as an adjunction between the categories of Rota-Baxter Hom-associative algebras and of Hom-(tri)dendriform algebras, are constructed.

1. INTRODUCTION

Rota-Baxter operators have appeared in a wide range of areas in pure and applied mathematics. The paradigmatic example of a Rota-Baxter operator concerns the integration by parts formula. The algebraic formulation of a Rota-Baxter algebra first appeared in G. Baxter's work in probability and the study of fluctuation theory [11]. This algebra was intensively studied by G.C. Rota [62, 63] in connection with combinatorics. In the work of A. Connes and D. Kreimer [20] about their Hopf algebra approach to renormalization of quantum field theory, the Rota-Baxter identity appeared under the name *multiplicativity constraint*. This seminal work gives rise to an important development including Rota-Baxter algebras and their connections to other algebraic structure. See [2, 4, 24, 25, 27, 28, 29, 30, 31, 32, 33, 38, 39, 40, 41, 42, 51, 10]. Rota-Baxter operator in the context of Lie algebras were introduced independently by Belavin and Drinfeld [12] and Semenov-Tian-Shansky [64]. Rota-Baxter Lie algebras are related to solutions of the (modified) classical Yang-Baxter equation.

Closely related to Rota-Baxter algebras are dendriform algebras, which were introduced by Loday in [54]. Dendriform algebras have two binary operations, which dichotomize an associative multiplication. The motivation to introduce these algebraic structures comes from K -theory. Dendriform algebras are connected to several areas in mathematics and physics, including Hopf algebras, homotopy Gerstenhaber algebra, operads, homology, combinatorics, and quantum field theory, where they occur in the theory of renormalization of Connes and Kreimer. Rota-Baxter algebras are related to dendriform algebras via a pair of adjoint functors [24, 26]. Roughly speaking, Rota-Baxter algebras are to dendriform algebras as associative algebras are to Lie algebras.

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. Hom-algebra structures first arose in quasi-deformations of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures, in which the Jacobi condition is twisted. The first examples of q -deformations, in which the derivations are replaced by σ -derivations, concerned the Witt and the Virasoro algebras; see for example [1, 16, 17, 18, 19, 21, 22, 47, 52, 45]. A general study and construction of Hom-Lie algebras are considered in [43, 48, 49]. A more general framework bordering color and super Lie algebras was introduced in [43, 48, 49, 50]. In the subclass of Hom-Lie algebras, skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

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Hom-associative algebras, which generalize associative algebras to a situation where associativity is twisted by a linear map, was introduced in [58]. The commutator bracket defined using the multiplication in a Hom-associative algebra leads naturally to a Hom-Lie algebra. This provides a different way of constructing Hom-Lie algebras. Also introduced in [58] are Hom-Lie-admissible algebras and more general G -Hom-associative algebras, which are twisted generalizations of Lie-admissible and G -associative algebras [37], respectively. The class of G -Hom-associative algebras includes the subclasses of Hom-Vinberg and Hom-preLie algebras, generalizing to the twisted situation Vinberg and preLie algebras, respectively. It was shown in [58] that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well.

Enveloping algebras of Hom-Lie algebras were discussed in [66, 69]. The fundamentals of the formal deformation theory and associated cohomology structures for Hom-Lie algebras have been considered initially in [60] and completed in [6]. Homology for Hom-Lie algebras was developed in [67]. In [59] and [61], the theory of Hom-coalgebras and related structures are developed. Further development could be found in, for example, [34, 56, 57, 7, 9, 56, 15, 46], [68]–[73], and the references therein.

The purpose of this paper is to study a common generalization of Rota-Baxter algebras and Hom-Lie-admissible algebras, called Rota-Baxter Hom-Lie-admissible algebras, and the closely related Hom-dendriform algebras. We explore their free algebras and the connections between their categories. As we will show in this paper, Rota-Baxter operators provide new ways of going between the various subclasses of Hom-Lie-admissible algebras.

The rest of this paper is organized as follows. We summarize in the next section the basics of Hom-Lie-admissible algebras. In Section 3, we introduce Rota-Baxter G -Hom-associative algebras and provide several construction results. In Section 4, it is shown that some Rota-Baxter Hom-Lie-admissible and Hom-associative algebras yield left Hom-preLie algebras. Free Rota-Baxter G -Hom-associative algebras are discussed in Section 5. The construction of the free Rota-Baxter G -Hom-associative algebra involves the combinatorial objects of decorated trees and is formally similar to the construction of the enveloping Hom-associative algebras of Hom-Lie algebras [67, 69]. Free Rota-Baxter algebras have been studied in [5, 25].

In Section 6, we discuss Hom-dendriform algebras, their associated Hom-associative and Hom-preLie algebras, and free Hom-dendriform algebras. The last section is dedicated to establishing functors between the categories of Rota-Baxter Hom-associative algebras and of Hom-(tri)dendriform algebras.

2. HOM-LIE ADMISSIBLE AND G -HOM-ASSOCIATIVE ALGEBRAS

In this Section we recall the main result of Hom-Lie-admissible algebras in [58] and summarize the definitions and some properties of G -Hom-associative algebras. The latter are twisted generalizations of the G -associative algebras introduced in [37].

2.1. Convention. Throughout this article we work over a fixed field \mathbb{K} of characteristic 0.

2.2. Hom-algebras. A *Hom-module* is a pair (A, α) consisting of a \mathbb{K} -module A and a linear self-map $\alpha: A \rightarrow A$, called the *twisting map*. A *morphism* of Hom-modules $f: (A, \alpha) \rightarrow (A', \alpha')$ is a linear map $f: A \rightarrow A'$ such that $f \circ \alpha = \alpha' \circ f$. The category of Hom-modules is denoted by **HomMod**.

A *Hom-algebra* is a triple (A, μ, α) consisting of a Hom-module (A, α) and a bilinear map $\mu: A \times A \rightarrow A$. Such a Hom-algebra is often denoted simply by A . A Hom-algebra (A, μ, α) is said to be *multiplicative* if for all $x, y \in A$ we have $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$.

Let (A, μ, α) and $A' = (A', \mu', \alpha')$ be two Hom-algebras. A *morphism* $f: A \rightarrow A'$ of Hom-algebras is a linear map such that $\mu' \circ (f \otimes f) = f \circ \mu$ and $f \circ \alpha = \alpha' \circ f$.

In particular, two Hom-algebras (A, μ, α) and (A, μ', α') are *isomorphic* if there exists a linear isomorphism f such that $\mu = f^{-1} \circ \mu' \circ (f \otimes f)$ and $\alpha = f^{-1} \circ \alpha' \circ f$.

2.3. Hom-Lie algebras. The notion of a Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [43, 48, 49], motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. In this article, we follow the notations and a slightly more general definition of a Hom-Lie algebra from [58].

Definition 2.4. A *Hom-Lie algebra* is a Hom-algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ that satisfies

$$(2.1) \quad [x_1, x_2] = -[x_2, x_1], \quad (\text{skew-symmetry})$$

$$(2.2) \quad \bigcirc_{x_1, x_2, x_3} [\alpha(x_1), [x_2, x_3]] = 0 \quad (\text{Hom-Jacobi identity})$$

for all x_1, x_2, x_3 in \mathfrak{g} , where \bigcirc_{x_1, x_2, x_3} denotes the summation over the cyclic permutations of x_1, x_2, x_3 .

When the twisting map α is the identity map, we recover classical Lie algebras. The Hom-Jacobi identity (2.2) is the Jacobi identity in this case.

The classical concept of Lie-admissible algebras (see for example [37]) is extended to the Hom-setting in [58] as follows:

Definition 2.5. A *Hom-Lie-admissible algebra* is a Hom-algebra (A, μ, α) in which the commutator bracket, defined for all $x_1, x_2 \in A$ by

$$[x_1, x_2] = \mu(x_1, x_2) - \mu(x_2, x_1),$$

satisfies the Hom-Jacobi identity (2.2).

Remark 2.6. The commutator bracket is automatically skew-symmetric. Thus, if it satisfies the Hom-Jacobi identity, then it defines a Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$. When (A, μ, α) is a Hom-Lie-admissible algebra, the Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$ is also called the *commutator Hom-Lie algebra*.

Let (A, μ, α) be a Hom-algebra. The *Hom-associator* of A is the trilinear map \mathfrak{as}_A on A defined for $x_1, x_2, x_3 \in A$ by

$$(2.3) \quad \mathfrak{as}_A(x_1, x_2, x_3) = \mu(\mu(x_1, x_2), \alpha(x_3)) - \mu(\alpha(x_1), \mu(x_2, x_3)).$$

When the twisting map α is the identity map, the Hom-associator reduces to the usual associator.

Definition 2.7. Let G be a subgroup of the permutation group S_3 . A *G-Hom-associative algebra* is a Hom-algebra (A, μ, α) that satisfies the identity

$$(2.4) \quad \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathfrak{as}_A(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0,$$

where the x_i are in A , and $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ .

The condition (2.4) is called the *G-Hom-associative identity*. It is equivalent to

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathfrak{as}_A \circ \sigma = 0,$$

where $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. When $\alpha = \text{Id}$, (2.4) is called the *G-associative identity* [37].

Remark 2.8. Suppose (A, μ, α) is a Hom-algebra, and $[x, y] = \mu(x, y) - \mu(y, x)$ denotes the commutator bracket. Then the Hom-algebra $(A, [\cdot, \cdot], \alpha)$ satisfies the Hom-Jacobi identity if and only if the S_3 -Hom-associative identity (2.4)

$$\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} \mathfrak{as}_A(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0$$

holds for all $x_i \in A$. In other words, Hom-Lie-admissible algebras are all S_3 -Hom-associative algebras. The following result gives the converse of this observation.

Proposition 2.9 ([58]). *Let G be a subgroup of the permutation group \mathcal{S}_3 . Then every G -Hom-associative algebra is a Hom-Lie-admissible algebra.*

The subgroups of \mathcal{S}_3 are

$$\begin{aligned} G_1 &= \{\text{Id}\}, \quad G_2 = \{\text{Id}, \sigma_{12}\}, \quad G_3 = \{\text{Id}, \sigma_{23}\}, \\ G_4 &= \{\text{Id}, \sigma_{13}\}, \quad G_5 = A_3, \quad G_6 = \mathcal{S}_3, \end{aligned}$$

where A_3 is the alternating group, and σ_{ij} is the transposition between i and j . In view of Proposition 2.9, it is natural to introduce the following subclasses of Hom-Lie-admissible algebras corresponding to the subgroups of \mathcal{S}_3 .

- The G_1 -Hom-associative algebras are also called *Hom-associative algebras* and satisfy the G_1 -Hom-associative identity

$$(2.5) \quad \mu(\alpha(x_1), \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), \alpha(x_3)),$$

or equivalently

$$\mathfrak{as}_A = 0.$$

When the twisting map α is the identity map, we recover an associative algebra.

- The G_2 -Hom-associative algebras are also called *left Hom-preLie algebras*, *Hom-Vinberg algebras*, and *left Hom-symmetric algebras*. They satisfy the G_2 -Hom-associative identity

$$(2.6) \quad \mu(\alpha(x_1), \mu(x_2, x_3)) - \mu(\alpha(x_2), \mu(x_1, x_3)) = \mu(\mu(x_1, x_2), \alpha(x_3)) - \mu(\mu(x_2, x_1), \alpha(x_3)),$$

which is equivalent to saying that the Hom-associator is symmetric in the first two variables. When the twisting map α is the identity map, we recover a left preLie algebra, also known as a Vinberg algebra and a left symmetric algebra.

- The G_3 -Hom-associative algebras are also called *right Hom-preLie algebras* and *right Hom-symmetric algebras*. They satisfy the G_3 -Hom-associative identity

$$\mu(\alpha(x_1), \mu(x_2, x_3)) - \mu(\alpha(x_1), \mu(x_3, x_2)) = \mu(\mu(x_1, x_2), \alpha(x_3)) - \mu(\mu(x_1, x_3), \alpha(x_2)),$$

which is equivalent to saying that the Hom-associator is symmetric in the last two variables. When the twisting map α is the identity map, we recover a right pre-Lie algebra, also known as a right symmetric algebra.

- The G_4 -Hom-associative algebras satisfy the G_4 -Hom-associative identity

$$\mu(\alpha(x_1), \mu(x_2, x_3)) - \mu(\alpha(x_3), \mu(x_2, x_1)) = \mu(\mu(x_1, x_2), \alpha(x_3)) - \mu(\mu(x_3, x_2), \alpha(x_1)).$$

- The G_5 -Hom-associative algebras satisfy the A_3 -Hom-associative identity

$$\begin{aligned} &\mu(\alpha(x_1), \mu(x_2, x_3)) + \mu(\alpha(x_2), \mu(x_3, x_1)) + \mu(\alpha(x_3), \mu(x_1, x_2)) = \\ &\mu(\mu(x_1, x_2), \alpha(x_3)) + \mu(\mu(x_2, x_3), \alpha(x_1)) + \mu(\mu(x_3, x_1), \alpha(x_2)). \end{aligned}$$

If the product μ is skew-symmetric, then the previous condition is exactly the Hom-Jacobi identity.

- The G_6 -Hom-associative algebras are exactly the Hom-Lie-admissible algebras.

Remark 2.10. A Hom-associative algebra is also a G -Hom-associative algebra for every subgroup G of \mathcal{S}_3 . A left Hom-preLie algebra is the opposite algebra of a right Hom-preLie algebra.

3. ROTA-BAXTER OPERATORS AND G -HOM-ASSOCIATIVE ALGEBRAS

In this section, we extend the notion of Rota-Baxter algebra to G -Hom-associative algebras and prove some construction results.

Definition 3.1. A *Rota-Baxter G -Hom-associative algebra* (of weight λ) is a G -Hom-associative algebra (A, \cdot, α) together with a linear self-map $R: A \rightarrow A$ that satisfies the identity

$$(3.1) \quad R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y),$$

where $\lambda \in \mathbb{K}$ is a fixed scalar called the *weight*. The map R is called a *Rota-Baxter operator*, and the identity (3.1) is called the *Rota-Baxter identity*. A *morphism* of Rota-Baxter G -Hom-associative algebras is a morphism of Hom-algebras that commutes with the Rota-Baxter operators. With G understood, the category of Rota-Baxter G -Hom-associative algebras (of weight λ) is denoted by \mathbf{HomRB}_λ . A Rota-Baxter G -Hom-associative algebra is *multiplicative* if the Hom-algebra (A, \cdot, α) is multiplicative and $\alpha R = R\alpha$.

When the weight λ is understood, we denote a Rota-Baxter G -Hom-associative algebra by a quadruple (A, \cdot, α, R) . We obtain *Rota-Baxter G -associative algebras* when the twisting map α is the identity map, and we denote them by triples (A, \cdot, R) . A Rota-Baxter G_1 -Hom-associative (resp., G_6 -Hom-associative) algebra will also be called a *Rota-Baxter Hom-associative* (resp., *Hom-Lie-admissible*) algebra.

Remark 3.2. A Rota-Baxter $\{\text{Id}\}$ -associative algebra is exactly what is usually called a Rota-Baxter algebra. A Rota-Baxter A_3 -associative algebra whose multiplication is skew-symmetric is exactly a Rota-Baxter Lie algebra.

Remark 3.3. A Rota-Baxter Hom-Lie algebra is a Rota-Baxter A_3 -Hom-associative algebra in which the product μ is skew-symmetric. Rota-Baxter Hom-Lie algebras are closely related to Hom-Novikov algebras; see [71, Theorem 1.4].

Example 3.4. Let (A, \cdot, α, R) be a Rota-Baxter G -Hom-associative algebra of weight λ . Then

$$(A, \cdot, \alpha, -R - \lambda \text{Id})$$

is a Rota-Baxter G -Hom-associative algebra of weight λ .

Example 3.5. Let (A, \cdot, α, R) be a Rota-Baxter G -Hom-associative algebra of weight λ . Then $(A, \cdot, \alpha, -R)$ is a Rota-Baxter G -Hom-associative algebra of weight $-\lambda$.

Example 3.6. Suppose $\lambda \in \mathbb{K}$ is a non-zero scalar. Then (A, \cdot, α, R) is a Rota-Baxter G -Hom-associative algebra of weight λ if and only if

$$(A, \cdot, \alpha, \lambda^{-1} R)$$

is a Rota-Baxter G -Hom-associative algebra of weight 1.

Example 3.7 (Jackson \mathfrak{sl}_2). The Jackson \mathfrak{sl}_2 , denoted $q\mathfrak{sl}_2$, is a q -deformation of the classical \mathfrak{sl}_2 . This family of Hom-Lie algebras was constructed in [50] using a quasi-deformation scheme based on discretizing by means of Jackson q -derivations a representation of $\mathfrak{sl}_2(\mathbb{K})$ by one-dimensional vector fields (first order ordinary differential operators) and using the twisted commutator bracket defined in [43]. It carries a Hom-Lie algebra structure but not a Lie algebra structure for $q \neq 1$. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that:

$$\begin{aligned} [x_1, x_2] &= -2qx_2, & \alpha(x_1) &= qx_1, \\ [x_1, x_3] &= 2x_3, & \alpha(x_2) &= q^2x_2, \\ [x_2, x_3] &= -\frac{1}{2}(1+q)x_1, & \alpha(x_3) &= qx_3, \end{aligned}$$

where q is a parameter in \mathbb{K} . If $q = 1$ we recover the classical \mathfrak{sl}_2 .

Let R_1 and R_2 be the following two operators defined on $q\mathfrak{sl}_2$ with respect to the basis $\{x_1, x_2, x_3\}$ by:

$$\begin{aligned} R_1(x_1) &= 0, & R_2(x_1) &= \rho_1 x_1 - \frac{(1+q)\rho_1^2}{\rho_2} x_3, \\ R_1(x_2) &= \rho_1 x_2 + q \frac{\rho_1^2}{\rho_2^2} x_3, & R_2(x_2) &= \frac{1}{2}\rho_1 x_2 + \frac{(1+q)^2 \rho_1^3}{8\rho_2^3} x_3, \\ R_1(x_3) &= \rho_2 x_2 + q\rho_1 x_3, & R_3(x_3) &= \rho_2 x_1 + \frac{2q\rho_2^2}{(1+q)^2 \rho_1} x_2 - \frac{(2+q)\rho_1}{2} x_3, \end{aligned}$$

where ρ_1, ρ_2 are parameters in \mathbb{K}^* and $q \neq -1$. Then $(q\mathfrak{sl}_2, [\cdot, \cdot], \alpha, R_1)$ and $(q\mathfrak{sl}_2, [\cdot, \cdot], \alpha, R_2)$ are Rota-Baxter Hom-Lie algebras of weight 0.

Example 3.8. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over \mathbb{K} . The following multiplication \cdot and linear map α on A define Hom-associative algebras over \mathbb{K}^3 :

$$\begin{aligned} x_1 \cdot x_1 &= a x_1, & x_2 \cdot x_2 &= a x_2, \\ x_1 \cdot x_2 &= x_2 \cdot x_1 = a x_2, & x_2 \cdot x_3 &= b x_3, \\ x_1 \cdot x_3 &= x_3 \cdot x_1 = b x_3, & x_3 \cdot x_2 &= x_3 \cdot x_3 = 0, \end{aligned}$$

$$\alpha(x_1) = a x_1, \quad \alpha(x_2) = a x_2, \quad \alpha(x_3) = b x_3,$$

where a, b are parameters in \mathbb{K} . The algebras are not associative when $a \neq b$ and $b \neq 0$, since

$$(x_1 \cdot x_1) \cdot x_3 - x_1 \cdot (x_1 \cdot x_3) = (a - b)bx_3.$$

Let R be the operator defined with respect to the basis $\{x_1, x_2, x_3\}$ by

$$R(x_1) = \rho_1 x_3, \quad R(x_2) = \rho_2 x_3, \quad R(x_3) = 0,$$

where ρ_1, ρ_2 are parameters in \mathbb{K} . Then (A, \cdot, α, R) is a Rota-Baxter Hom-associative algebra of weight 0.

In the rest of this Section, we apply the twisting principles, introduced in [67, Theorem 2.4] and [73, Theorem 2.11], to obtain examples of Rota-Baxter G -Hom-associative algebras. Also, we extend to Rota-Baxter G -associative algebras the construction involving elements of the centroid used in [13, Proposition 1.12].

The following result states that a Rota-Baxter G -Hom-associative algebra yields another Rota-Baxter G -Hom-associative algebra when its multiplication and twisting map are twisted by a morphism.

Theorem 3.9. *Let (A, μ, α, R) be a Rota-Baxter G -Hom-associative algebra of weight λ and $\beta: A \rightarrow A$ be a morphism. Then*

$$A_\beta = (A, \mu_\beta = \beta\mu, \beta\alpha, R)$$

is also a Rota-Baxter G -Hom-associative algebra of weight λ . Moreover, if A is multiplicative, then so is A_β .

Furthermore, suppose (A', μ', α', R') is a Rota-Baxter G -Hom-associative algebra and $\beta': A' \rightarrow A'$ is a morphism. If $f: A \rightarrow A'$ is a morphism such that $f\beta = \beta'f$, then

$$f: A_\beta \rightarrow A'_{\beta'}$$

is also a morphism.

Proof. To see that A_β is a G -Hom-associative algebra, note that the Hom-associators of A and A_β are related as

$$\mathbf{as}_{A_\beta} = \beta^2 \circ \mathbf{as}_A.$$

Thus, the G -Hom-associative identity in A implies

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathbf{as}_{A_\beta} \circ \sigma = \beta^2 \circ \left(\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathbf{as}_A \circ \sigma \right) = 0.$$

This shows that A_β is a G -Hom-associative algebra.

The Rota-Baxter identity (3.1) for A_β is true by the following commutative diagram:

$$(3.2) \quad \begin{array}{ccccccc} A^{\otimes 2} & \xrightarrow{\chi} & A^{\otimes 2} & \xrightarrow{\mu} & A & \xrightarrow{\beta} & A \\ \downarrow R^{\otimes 2} & & & & \downarrow R & & \downarrow R \\ A^{\otimes 2} & \xrightarrow{\mu} & A & \xrightarrow{\beta} & A & & A \end{array}$$

Here

$$\chi = R \otimes \text{Id} + \text{Id} \otimes R + \lambda(\text{Id}^{\otimes 2}),$$

so the left square is commutative by the Rota-Baxter identity in A . The right rectangle is commutative because β is a morphism. We have shown that A_β is a Rota-Baxter G -Hom-associative algebra.

The assertions concerning multiplicativity and the morphism f are obvious. \square

Let us discuss two special cases of Theorem 3.9. The following result says that each multiplicative Rota-Baxter G -Hom-associative algebra yields a sequence of multiplicative Rota-Baxter G -Hom-associative algebras with twisted multiplications and twisting maps.

Corollary 3.10. *Let (A, μ, α, R) be a multiplicative Rota-Baxter G -Hom-associative algebra of weight λ . Then*

$$A_{\alpha^n} = (A, \alpha^n \mu, \alpha^{n+1}, R)$$

is also a multiplicative Rota-Baxter G -Hom-associative algebra of weight λ for each $n \geq 1$.

Proof. This is the $\beta = \alpha^n$ special case of Theorem 3.9. \square

The following result says that a Rota-Baxter G -associative algebra deforms into a multiplicative Rota-Baxter G -Hom-associative algebra via a morphism.

Corollary 3.11. *Let (A, μ, R) be a Rota-Baxter G -associative algebra of weight λ and $\beta: A \rightarrow A$ be a morphism. Then*

$$A_\beta = (A, \mu_\beta = \beta \mu, \beta, R)$$

is a multiplicative Rota-Baxter G -Hom-associative algebra of weight λ .

Proof. This is the $\alpha = \text{Id}$ special case of Theorem 3.9. \square

As a converse to Corollary 3.11, given a multiplicative Rota-Baxter G -Hom-associative algebra (A, μ, α, R) , one may ask whether it is induced by an ordinary Rota-Baxter G -associative algebra $(A, \tilde{\mu}, R)$. In other words, we ask whether there exists a morphism α on $(A, \tilde{\mu}, R)$ such that $\mu = \alpha \circ \tilde{\mu}$.

Let (A, μ, α) be a multiplicative G -Hom-associative algebra. Following the observation in [36], when α is invertible, the twisting principle with α^{-1} leads to a G -associative algebra. If α is an algebra morphism with respect to $\tilde{\mu}$, then α is also an algebra morphism with respect to $\mu = \alpha \circ \tilde{\mu}$ because

$$\mu(\alpha(x), \alpha(y)) = \alpha \circ \tilde{\mu}(\alpha(x), \alpha(y)) = \alpha \circ \alpha \circ \tilde{\mu}(x, y) = \alpha \circ \mu(x, y).$$

If α is bijective, then α^{-1} is also an algebra automorphism. Therefore, one may use an untwist operation on the G -Hom-associative algebra to recover the G -associative algebra ($\tilde{\mu} = \alpha^{-1} \circ \mu$).

The following result is a partial converse to Corollary 3.11.

Proposition 3.12. *Let (A, μ, α, R) be a multiplicative Rota-Baxter G -Hom-associative algebra of weight λ in which α is invertible and commutes with R . Then*

$$A' = (A, \mu_{\alpha^{-1}} = \alpha^{-1} \circ \mu, R)$$

is a Rota-Baxter G -associative algebra of weight λ .

Proof. The associator of A' and the Hom-associator of A are related as

$$\text{as}_{A'} = \alpha^{-2} \circ \text{as}_A.$$

The G -Hom-associative identity in A now implies

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \text{as}_{A'} \circ \sigma = \alpha^{-2} \left(\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \text{as}_A \circ \sigma \right) = 0.$$

Therefore, the G -associative identity holds in A' .

Since α commutes with R , so does α^{-1} . Hence R is a Rota-Baxter operator for the multiplication $\mu_{\alpha^{-1}}$. \square

Next we construct Rota-Baxter G -Hom-associative algebras involving elements of the centroid of Rota-Baxter G -associative algebras. The construction of Hom-algebras using elements of the centroid was initiated in [13] for Lie algebras.

Let (A, μ, β) be a Hom-algebra. The *centroid* $\text{Cent}(A)$ of A is defined as the set consisting of linear self-maps $\alpha: A \rightarrow A$ satisfying the condition

$$\alpha(\mu(x, y)) = \mu(\alpha(x), y) = \mu(x, \alpha(y))$$

for all $x, y \in A$. Notice that if $\alpha \in \text{Cent}(A)$, then we have

$$\mu(\alpha^p(x), \alpha^q(y)) = \alpha^{p+q}\mu(x, y)$$

for all $p, q \geq 0$.

Theorem 3.13. *Let (A, μ, R) be a Rota-Baxter G -associative algebra of weight λ . Suppose $\alpha \in \text{Cent}(A)$ such that α and R commute. Then*

$$A(n, m) = (A, \mu_{\alpha^n} = \alpha^n \mu, \alpha^m, R)$$

is a Rota-Baxter G -Hom-associative algebra of weight λ for any $n, m \geq 0$.

Proof. The assumption that α lies in the centroid of A implies that the associator of A and the Hom-associator of $A(n, m)$ are related as

$$\mathbf{as}_{A(n, m)} = \alpha^{2n+m} \circ \mathbf{as}_A.$$

Thus, the G -associative identity in A implies

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathbf{as}_{A(n, m)} \circ \sigma = \alpha^{2n+m} \left(\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathbf{as}_A \circ \sigma \right) = 0.$$

Therefore, $A(n, m)$ is a G -Hom-associative algebra.

To see that the Rota-Baxter identity holds in $A(n, m)$, we can reuse the diagram (3.2) with $\beta = \alpha^n$. Once again the left square is commutative by the Rota-Baxter identity in A . The right rectangle is commutative because α commutes with R , and hence so does α^n . \square

4. FROM ROTA-BAXTER HOM-ALGEBRAS TO HOM-PRELIE ALGEBRAS

In this section, we construct a functor from a full subcategory of the category of Rota-Baxter Hom-Lie-admissible (or Hom-associative) algebras to the category of left Hom-preLie algebras.

Theorem 4.1. *Let (A, \cdot, α, R) be a Rota-Baxter Hom-Lie-admissible algebra of weight 0 such that α and R commute. Define the binary operation $*$ on A by*

$$(4.1) \quad x * y = R(x) \cdot y - y \cdot R(x) = [R(x), y].$$

Then

$$A_1 = (A, *, \alpha)$$

is a left Hom-preLie algebra.

Proof. For $x, y, z \in A$, we have:

$$\begin{aligned} \alpha(x) * (y * z) &= \alpha(x) * (R(y) \cdot z - z \cdot R(y)) \\ &= R(\alpha(x)) \cdot (R(y) \cdot z - z \cdot R(y)) - (R(y) \cdot z - z \cdot R(y)) \cdot R(\alpha(x)) \\ &= \alpha(R(x)) \cdot (R(y) \cdot z) - \alpha(R(x)) \cdot (z \cdot R(y)) \\ &\quad - (R(y) \cdot z) \cdot \alpha(R(x)) + (z \cdot R(y)) \cdot \alpha(R(x)), \end{aligned}$$

and

$$\begin{aligned}
 (x * y) * \alpha(z) &= (R(x) \cdot y - y \cdot R(x)) * \alpha(z) \\
 &= R(R(x) \cdot y - y \cdot R(x)) \cdot \alpha(z) - \alpha(z) \cdot R(R(x) \cdot y - y \cdot R(x)) \\
 &= R(R(x) \cdot y) \cdot \alpha(z) - R(y \cdot R(x)) \cdot \alpha(z) \\
 &\quad - \alpha(z) \cdot R(R(x) \cdot y) + \alpha(z) \cdot R(y \cdot R(x)).
 \end{aligned}$$

Subtracting the above terms, switching x and y , and then subtracting the result yield:

$$\begin{aligned}
 \mathfrak{as}_{A_1}(y, x, z) - \mathfrak{as}_{A_1}(x, y, z) &= \alpha(x) * (y * z) - (x * y) * \alpha(z) - \alpha(y) * (x * z) + (y * x) * \alpha(z) \\
 &= \underbrace{\alpha(R(x)) \cdot (R(y) \cdot z) - \alpha(R(x)) \cdot (z \cdot R(y)) - (R(y) \cdot z) \cdot \alpha(R(x)) + (z \cdot R(y)) \cdot \alpha(R(x))}_{[\alpha(R(x)), [R(y), z]]} \\
 &\quad - R(R(x) \cdot y) \cdot \alpha(z) + R(y \cdot R(x)) \cdot \alpha(z) + \alpha(z) \cdot R(R(x) \cdot y) - \alpha(z) \cdot R(y \cdot R(x)) \\
 &\quad - \underbrace{\alpha(R(y)) \cdot (R(x) \cdot z) + \alpha(R(y)) \cdot (z \cdot R(x)) + (R(x) \cdot z) \cdot \alpha(R(y)) - (z \cdot R(x)) \cdot \alpha(R(y))}_{[\alpha(R(y)), [z, R(x)]]} \\
 &\quad + R(R(y) \cdot x) \cdot \alpha(z) - R(x \cdot R(y)) \cdot \alpha(z) - \alpha(z) \cdot R(R(y) \cdot x) + \alpha(z) \cdot R(x \cdot R(y)).
 \end{aligned}$$

Gathering the 5th and 14th, 6th and 13th, 7th and 16th, and 8th and 15th terms, and using the Rota-Baxter identity (3.1) with $\lambda = 0$, we obtain:

$$\begin{aligned}
 \mathfrak{as}_{A_1}(y, x, z) - \mathfrak{as}_{A_1}(x, y, z) &= [\alpha(R(x)), [R(y), z]] + [\alpha(R(y)), [z, R(x)]] \\
 &\quad - (R(x) \cdot R(y)) \cdot \alpha(z) + (R(y) \cdot R(x)) \cdot \alpha(z) + \alpha(z) \cdot (R(x) \cdot R(y)) - \alpha(z) \cdot (R(y) \cdot R(x)) \\
 &= [\alpha(R(x)), [R(y), z]] + [\alpha(z), [R(x), R(y)]] + [\alpha(R(y)), [z, R(x)]] \\
 &= \circlearrowleft_{R(x), R(y), z} [\alpha(R(x)), [R(y), z]].
 \end{aligned}$$

Hom-Lie-admissibility now implies

$$\mathfrak{as}_{A_1}(y, x, z) - \mathfrak{as}_{A_1}(x, y, z) = 0,$$

showing that A_1 is a left Hom-preLie algebra. \square

Theorem 4.2. *Let (A, \cdot, α, R) be a Rota-Baxter Hom-associative algebra of weight -1 such that α and R commute. Define the operation $*$ on A by*

$$\begin{aligned}
 (4.2) \quad x * y &= R(x) \cdot y - y \cdot R(x) - x \cdot y \\
 &= [R(x), y] - x \cdot y.
 \end{aligned}$$

Then

$$A_2 = (A, *, \alpha)$$

is a left Hom-preLie algebra.

Proof. For $x, y, z \in A$, we have:

$$\begin{aligned}
 \alpha(x) * (y * z) &= R(\alpha(x)) \cdot (R(y) \cdot z - z \cdot R(y) - y \cdot z) \\
 &\quad - (R(y) \cdot z - z \cdot R(y) - y \cdot z) \cdot R(\alpha(x)) \\
 &\quad - \alpha(x) \cdot (R(y) \cdot z - z \cdot R(y) - y \cdot z),
 \end{aligned}$$

and

$$\begin{aligned}
 (x * y) * \alpha(z) &= R(R(x) \cdot y - y \cdot R(x) - x \cdot y) \cdot \alpha(z) \\
 &\quad - \alpha(z) \cdot R(R(x) \cdot y - y \cdot R(x) - x \cdot y), \\
 &\quad - (R(x) \cdot y - y \cdot R(x) - x \cdot y) \cdot \alpha(z).
 \end{aligned}$$

Then using the fact that α and R commute, and Hom-associativity we obtain:

$$\begin{aligned}
& \mathbf{as}_{A_2}(y, x, z) - \mathbf{as}_{A_2}(x, y, z) \\
&= \alpha(x) * (y * z) - (x * y) * \alpha(z) - \alpha(y) * (x * z) + (y * x) * \alpha(z) \\
&= \alpha(R(x)) \cdot (R(y) \cdot z) + (z \cdot R(y)) \cdot \alpha(R(x)) \\
&\quad - \alpha(R(y)) \cdot (R(x) \cdot z) - (z \cdot R(x)) \cdot \alpha(R(y)) \\
&\quad - R(R(x) \cdot y) \cdot \alpha(z) + R(y \cdot R(x)) \cdot \alpha(z) \\
&\quad + R(R(y) \cdot x) \cdot \alpha(z) - R(x \cdot R(y)) \cdot \alpha(z) \\
&\quad + \alpha(z) \cdot R(R(x) \cdot y) - \alpha(z) \cdot R(y \cdot R(x)) \\
&\quad - \alpha(z) \cdot R(R(y) \cdot x) + \alpha(z) \cdot R(x \cdot R(y)) \\
&\quad + R(x \cdot y) \cdot \alpha(z) - \alpha(z) \cdot R(x \cdot y) \\
&\quad - R(y \cdot x) \cdot \alpha(z) + \alpha(z) \cdot R(y \cdot x).
\end{aligned}$$

The above sum vanishes by Hom-associativity and the Rota-Baxter identity (3.1) with $\lambda = -1$. \square

5. FREE ROTA-BAXTER G -HOM-ASSOCIATIVE ALGEBRAS

The purpose of this section is to give an explicit construction of the free Rota-Baxter G -Hom-associative algebra of weight λ associated to a Hom-module. Recall that \mathbf{HomRB}_λ and \mathbf{HomMod} denote the categories of Rota-Baxter G -Hom-associative algebras of weight λ and of Hom-modules, respectively. Here is the main result of this section.

Theorem 5.1. *The forgetful functor*

$$\mathcal{O}: \mathbf{HomRB}_\lambda \rightarrow \mathbf{HomMod}$$

given by

$$\mathcal{O}(A, \mu, \alpha, R) = (A, \alpha)$$

admits a left adjoint.

Proof. We consider an intermediate category \mathbf{D} , which we will make precise below. There are forgetful functors

$$(5.1) \quad \mathbf{HomRB}_\lambda \xrightarrow{\mathcal{O}_2} \mathbf{D} \xrightarrow{\mathcal{O}_1} \mathbf{HomMod},$$

whose composition is \mathcal{O} . We will show that each of these two forgetful functors \mathcal{O}_i admits a left adjoint \mathcal{F}_i (Theorems 5.3 and 5.8). The composition

$$\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$$

is then the desired left adjoint in Theorem 5.1. It remains to prove Theorems 5.3 and 5.8. \square

We begin by defining the intermediate category \mathbf{D} .

5.2. The category \mathbf{D} . Let \mathbf{D} be the category whose objects are quadruples (A, μ, α, R) in which:

- (1) A is a \mathbb{K} -module,
- (2) $\mu: A^{\otimes 2} \rightarrow A$ is a bilinear operation, and
- (3) α and R are linear self-maps $A \rightarrow A$.

A morphism $f: (A, \mu_A, \alpha_A, R_A) \rightarrow (B, \mu_B, \alpha_B, R_B)$ in \mathbf{D} consists of a linear map $f: A \rightarrow B$ such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$, $f \circ \alpha_A = \alpha_B \circ f$, and $f \circ R_A = R_B \circ f$.

There is a forgetful functor

$$(5.2) \quad \mathcal{O}_1: \mathbf{D} \rightarrow \mathbf{HomMod}$$

defined as

$$\mathcal{O}_1(A, \mu, \alpha, R) = (A, \alpha).$$

Theorem 5.3. *The functor \mathcal{O}_1 in (5.2) admits a left adjoint*

$$\mathcal{F}_1: \mathbf{HomMod} \rightarrow \mathbf{D}.$$

In other words, the functor \mathcal{F}_1 associates to a Hom-module its free object in \mathbf{D} . In order to construct the free functor \mathcal{F}_1 , we need to freely generate products (for μ) and images of R , while keeping α defined. This involves an elaborate process of bookkeeping, for which we use the notion of decorated trees. The proof of Theorem 5.3 will be given after the following preliminary discussions of trees.

5.4. Planar binary trees. For $n \geq 1$, let T_n denote the set of (isomorphism classes of) planar binary trees with n leaves and one root. The first few T_n are depicted below.

$$T_1 = \left\{ \begin{array}{c} \bullet \\ | \end{array} \right\}, \quad T_2 = \left\{ \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right\}, \quad T_3 = \left\{ \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \right\}, \quad T_4 = \left\{ \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagdown \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet \end{array} \right\}.$$

Each node represents either a leaf, which is always depicted at the top, or an internal vertex. An element in T_n will be called an n -tree. The set of nodes (= leaves and internal vertices) in a tree ψ is denoted by $N(\psi)$. The node of an n -tree ψ that is connected to the root (the lowest point in the n -tree) will be denoted by v_{low} . In other words, v_{low} is the lowest internal vertex in ψ if $n \geq 2$ and is the only leaf if $n = 1$.

Given an n -tree ψ and an m -tree φ , their **grafting**

$$\psi \vee \varphi \in T_{n+m}$$

is the tree obtained by placing ψ on the left and φ on the right and joining their roots to form the new lowest internal vertex, which is connected to the new root. Pictorially, we have

$$\psi \vee \varphi = \begin{array}{c} \psi \quad \varphi \\ \diagdown \quad \diagup \\ \bullet \end{array}.$$

Note that grafting is a nonassociative operation. As we will discuss below, the operation of grafting is for generating the multiplication μ .

Conversely, by cutting the two upward branches from the lowest internal vertex, each n -tree ψ with $n \geq 2$ can be uniquely represented as the grafting of two trees, say, $\psi_1 \in T_p$ and $\psi_2 \in T_q$, where $p + q = n$. By iterating the grafting operation, one can show by a simple induction argument that every n -tree ($n \geq 2$) can be obtained as an iterated grafting of n copies of the 1-tree.

5.5. Decorated trees. By a **decorated n -tree**, we mean a pair (ψ, f) consisting of an n -tree ψ and a function f from the set of nodes $N(\psi)$ to the set of non-empty finite sequences of non-negative integers, satisfying the following two conditions.

(1) For any $v \in N(\psi)$, if

$$(5.3) \quad f(v) = (a_k, a_{k-1}, \dots, a_2, a_1),$$

then $a_j > 0$ for all $j \geq 2$.

(2) If $v \in N(\psi)$ is a leaf and $f(v)$ is as in (5.3), then $a_1 = 0$ implies $k = 1$.

The set of decorated n -trees is denoted by \overline{T}_n . For $v \in N(\psi)$, the finite sequence $f(v)$ is called the **decoration of v** . For example, here is a decorated 3-tree:

$$(5.4) \quad \begin{array}{c} (3, 5, 2) \quad (0) \quad (2, 6) \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array}$$

The decoration of each node is depicted right next to or on top of it.

As we will discuss below in (5.7), decorated n -trees give us a way of composing n elements in a Rota-Baxter G -Hom-associative algebra or, more generally, in an object in \mathbf{D} . Here let us give a heuristic explanation. The decoration

$$f(v) = (a_k, a_{k-1}, \dots, a_2, a_1)$$

of a node v is for generating the operation

$$\dots \alpha^{a_4} R^{a_3} \alpha^{a_2} R^{a_1}.$$

The requirement $a_j > 0$ for $j \geq 2$ in (5.3) is imposed because we do not need to generate the identity operation $\alpha^0 = R^0$. The other requirement, that $a_1 = 0$ implies $k = 1$ in (5.3) for a leaf, is imposed because a Hom-module (M, α) already has the operations α^n for all $n \geq 1$. Thus, we begin by generating the operations R^n for $n \geq 1$.

Given a decorated n -tree (ψ, f) and elements x_1, \dots, x_n in an object A in \mathbf{D} , imagine placing x_i at the i th leaf, counting from left to right, of ψ . The decoration (\dots, a_2, a_1) at the i th leaf tells us that we should apply $\dots R^{a_3} \alpha^{a_2} R^{a_1}$ to x_i . Once this is done to each of the x_i , we move downward along the branches. At each internal node, we first multiply the two elements coming from the two branches above the node. Then we apply $\dots R^{b_3} \alpha^{b_2} R^{b_1}$ to the resulting product according to the decoration (\dots, b_2, b_1) of that node. Once we have done this for the lowest internal node, we have an element in A .

5.6. Operations on decorated trees. In order to parametrize n -ary operations in an object in \mathbf{D} , we need operations on decorated trees corresponding to μ , R , and α , which we now discuss.

If $(\psi, f) \in \overline{T}_n$ and $(\varphi, g) \in \overline{T}_m$ are decorated trees, then their **grafting** is the $(n+m)$ -tree $\psi \vee \varphi$ with decoration

$$h(v) = \begin{cases} f(v) & \text{if } v \in N(\psi), \\ g(v) & \text{if } v \in N(\varphi), \\ (0) & \text{if } v = v_{low} \text{ in } \psi \vee \varphi. \end{cases}$$

For each $n \geq 1$, define a function $R: \overline{T}_n \rightarrow \overline{T}_n$ by setting

$$R(\psi, f) = (\psi, Rf),$$

where Rf is equal to f , except that the decoration

$$f(v_{low}) = (a_k, a_{k-1}, \dots, a_1)$$

of the lowest internal vertex (or the only leaf if $\psi \in T_1$) is replaced by

$$(5.5) \quad (Rf)(v_{low}) = \begin{cases} (a_k + 1, a_{k-1}, \dots, a_1) & \text{if } k \text{ is odd,} \\ (1, a_k, a_{k-1}, \dots, a_1) & \text{if } k \text{ is even} \end{cases}$$

Denote by $(1, (a_1))$ the 1-tree with decoration a_1 for its only leaf. Define the functions

$$\alpha: \begin{cases} \overline{T}_1 \setminus \{(1, (0))\} \rightarrow \overline{T}_1 \setminus \{(1, (0))\}, \\ \overline{T}_n \rightarrow \overline{T}_n \end{cases} \quad \text{for } n \geq 2$$

by setting

$$\alpha(\psi, f) = (\psi, \alpha f),$$

where αf is equal to f , except that

$$(5.6) \quad (\alpha f)(v_{low}) = \begin{cases} (1, a_k, a_{k-1}, \dots, a_1) & \text{if } k \text{ is odd,} \\ (a_k + 1, a_{k-1}, \dots, a_1) & \text{if } k \text{ is even.} \end{cases}$$

We emphasize that $\alpha(1, (0))$ is not defined.

Note that a simple induction argument shows that every decorated n -tree (ψ, f) can be obtained uniquely from n copies of $(1, (0))$ by iterating the operations \vee , α , and R .

5.7. Decorated trees acting on \mathbf{D} . Let (B, μ, α_B, R_B) be an object in \mathbf{D} . For each $n \geq 1$, define a map

$$(5.7) \quad \gamma: \mathbb{K}[\overline{T}_n] \otimes B^{\otimes n} \rightarrow B$$

as follows. Pick elements $\tau \in \overline{T}_n$, $\sigma \in \overline{T}_m$, and $b_1, b_2, \dots \in B$. Then we set

$$\gamma((1, (0)); b_1) = b_1,$$

and, inductively,

$$\begin{aligned} \gamma(\alpha\tau; b_1 \otimes \dots \otimes b_n) &= \alpha_B(\gamma(\tau; b_1 \otimes \dots \otimes b_n)), \\ \gamma(R\tau; b_1 \otimes \dots \otimes b_n) &= R_B(\gamma(\tau; b_1 \otimes \dots \otimes b_n)), \\ \gamma(\tau \vee \sigma; b_1 \otimes \dots \otimes b_{n+m}) &= \mu(\gamma(\tau; b_1 \otimes \dots \otimes b_n), \gamma(\sigma; b_{n+1} \otimes \dots \otimes b_{n+m})). \end{aligned}$$

For example, for the decorated 3-tree τ in (5.4), we have

$$\gamma(\tau; b_1 \otimes b_2 \otimes b_3) = R\alpha^8 \{ [\alpha^7 R^4 \alpha^9 R^2 ((R^3 \alpha^5 R^2(b_1)) \cdot b_2)] \cdot [\alpha^2 R^6(b_3)] \}.$$

Here, and in what follows, we write $\mu(x, y)$ as $x \cdot y$ or even xy .

Proof of Theorem 5.3. Pick a Hom-module (A, α_A) . Consider the \mathbb{K} -module

$$(5.8) \quad \mathcal{F}_1(A) = \bigoplus_{n \geq 1, \tau \in \overline{T}_n} A_\tau^{\otimes n},$$

where each $A_\tau^{\otimes n}$ is a copy of $A^{\otimes n}$. Identify A as a submodule of $\mathcal{F}_1(A)$ via the inclusion

$$(5.9) \quad \iota: A \xrightarrow{\cong} A_{(1, (0))} \hookrightarrow \mathcal{F}_1(A).$$

A typical generator in $A_\tau^{\otimes n}$ is denoted by $(a_1 \otimes \dots \otimes a_n)_\tau$. We claim that \mathcal{F}_1 is the desired left adjoint of \mathcal{O}_1 . First, for $\mathcal{F}_1(A)$ to be an object in \mathbf{D} , we need to equip $\mathcal{F}_1(A)$ with the three operations μ , α , and R . We make use of the operations on decorated trees discussed in section 5.6

The bilinear operation

$$\mu: \mathcal{F}_1(A) \otimes \mathcal{F}_1(A) \rightarrow \mathcal{F}_1(A)$$

is defined as

$$(5.10) \quad \mu((a_1 \otimes \dots \otimes a_n)_\tau, (a_{n+1} \otimes \dots \otimes a_{n+m})_\sigma) = (a_1 \otimes \dots \otimes a_{n+m})_{\tau \vee \sigma}$$

on the generators. The linear operations α and R on $\mathcal{F}_1(A)$ are defined as

$$(5.11) \quad \begin{aligned} \alpha((a_1 \otimes \dots \otimes a_n)_\tau) &= \begin{cases} (a_1 \otimes \dots \otimes a_n)_{\alpha(\tau)} & \text{if } \tau \neq (1, (0)), \\ (\alpha_A(a_1))_{(1, (0))} & \text{if } \tau = (1, (0)), \end{cases} \\ R((a_1 \otimes \dots \otimes a_n)_\tau) &= (a_1 \otimes \dots \otimes a_n)_{R(\tau)}. \end{aligned}$$

With these operations, $\mathcal{F}_1(A)$ becomes an object in \mathbf{D} . It is clear how \mathcal{F}_1 is defined on morphisms of Hom-modules and that \mathcal{F}_1 defines a functor $\mathbf{HomMod} \rightarrow \mathbf{D}$. Moreover, ι extends to a morphism

$$\iota: A \rightarrow \mathcal{F}_1(A)$$

of Hom-modules.

To show that \mathcal{F}_1 is the left adjoint of the forgetful functor \mathcal{O}_1 , pick an object $(B, \mu_B, \alpha_B, R_B) \in \mathbf{D}$. Let $f: (A, \alpha_A) \rightarrow (B, \alpha_B)$ be a morphism of Hom-modules. We need to show that there exists a unique morphism

$$\varphi: \mathcal{F}_1(A) \rightarrow B \in \mathbf{D}$$

such that

$$(5.12) \quad f = \varphi \circ \iota.$$

Define $\varphi: \mathcal{F}_1(A) \rightarrow B$ by setting

$$(5.13) \quad \varphi((a_1 \otimes \dots \otimes a_n)_\tau) = \gamma(\tau; f(a_1) \otimes \dots \otimes f(a_n))$$

on the generators and extending linearly, where γ is defined in (5.7). It is clear that (5.12) is satisfied.

It remains to show that φ is a morphism in \mathbf{D} . Suppose that $\tau \neq (1, (0))$. To show that φ is compatible with α , we compute as follows:

$$\begin{aligned}\varphi(\alpha(a_1 \otimes \cdots \otimes a_n)_\tau) &= \varphi((a_1 \otimes \cdots \otimes a_n)_{\alpha(\tau)}) \\ &= \gamma(\alpha(\tau); f(a_1) \otimes \cdots \otimes f(a_n)) \\ &= \alpha_B(\gamma(\tau; f(a_1) \otimes \cdots \otimes f(a_n))) \\ &= \alpha_B(\varphi((a_1 \otimes \cdots \otimes a_n)_\tau)).\end{aligned}$$

Replacing α by R , the same argument, regardless of whether $\tau = (1, (0))$ or not, also shows that φ is compatible with R . Now if $\tau = (1, (0))$, then we have

$$\begin{aligned}\varphi(\alpha((a)_{(1, (0))})) &= \varphi((\alpha_A(a))_{(1, (0))}) \\ &= \gamma((1, (0)); f(\alpha_A(a))) \\ &= f(\alpha_A(a)) \\ &= \alpha_B(f(a)) \\ &= \alpha_B \gamma((1, (0)); f(a)) \\ &= \alpha_B \varphi((a)_{(1, (0))}).\end{aligned}$$

This shows that φ is compatible with α .

Likewise, to show that φ is compatible with μ , we compute as follows:

$$\begin{aligned}\varphi(\mu((a_1 \otimes \cdots \otimes a_n)_\tau, (a_{n+1} \otimes \cdots \otimes a_{n+m})_\sigma)) &= \varphi((a_1 \otimes \cdots \otimes a_{n+m})_{\tau \vee \sigma}) \\ &= \gamma(\tau \vee \sigma; f(a_1) \otimes \cdots \otimes f(a_{n+m})) \\ &= \mu_B(\gamma(\tau; f(a_1) \otimes \cdots \otimes f(a_n)), \gamma(\sigma; f(a_{n+1}) \otimes \cdots \otimes f(a_{n+m}))) \\ &= \mu_B(\varphi((a_1 \otimes \cdots \otimes a_n)_\tau), \varphi((a_{n+1} \otimes \cdots \otimes a_{n+m})_\sigma)).\end{aligned}$$

This shows that $\varphi: \mathcal{F}_1(A) \rightarrow B$ is a morphism in \mathbf{D} .

The uniqueness of φ is clear. Indeed, suppose that

$$\phi: \mathcal{F}_1(A) \rightarrow B \in \mathbf{D}$$

is another morphism such that

$$f = \phi \circ \iota.$$

The definitions (5.10) and (5.11) and the fact that ϕ is compatible with μ , α , and R together imply that ϕ is uniquely determined by $\phi((a)_{(1, (0))})$, i.e., its restriction to A . But the restriction of ϕ to A is equal to f , so ϕ must be equal to φ . \square

To finish the proof of Theorem 5.1, consider the forgetful functor

$$(5.14) \quad \mathcal{O}_2: \mathbf{HomRB}_\lambda \rightarrow \mathbf{D}$$

that simply forgets the extra axioms satisfied by μ , α , and R . This forgetful functor is a full and faithful embedding. In particular, we identify \mathbf{HomRB}_λ as a full subcategory of \mathbf{D} using \mathcal{O}_2 .

Theorem 5.8. *The category \mathbf{HomRB}_λ is a reflective subcategory of \mathbf{D} , i.e., the functor \mathcal{O}_2 in (5.14) admits a left adjoint*

$$\mathcal{F}_2: \mathbf{D} \rightarrow \mathbf{HomRB}_\lambda.$$

In order to prove Theorem 5.8, we need to introduce ideals and quotients for objects in \mathbf{D} .

5.9. Ideals and quotients in \mathbf{D} . Let (A, μ, α, R) be an object in \mathbf{D} . An *ideal in A* is a submodule $I \subseteq A$ that is closed under both α and R and such that

$$\mu(I, A) \subseteq I \quad \text{and} \quad \mu(A, I) \subseteq I.$$

In this case, the quotient module A/I is naturally an object in \mathbf{D} in which μ , α , and R are induced by those in A .

For example, if $f: A \rightarrow B$ is a morphism in \mathbf{D} , then the kernel of f is an ideal in A .

Proposition 5.10. *Let (A, μ, α, R) be an object in \mathbf{D} and S be a non-empty subset in A . Then there exists a unique ideal $\langle S \rangle$ in A such that*

- (1) $S \subseteq \langle S \rangle$, and
- (2) if I is an ideal in A that contains S , then I also contains $\langle S \rangle$.

The ideal $\langle S \rangle$ is called the *ideal generated by S* .

Proof. Given elements $x_1, \dots, x_n \in A$, by a *parenthesized monomial*

$$x_1 \cdots x_n \in A,$$

we mean any possible way of multiplying the x_i 's using μ in the prescribed order. For example, there is only one parenthesized monomial $x_1 x_2$. There are two parenthesized monomials $x_1 x_2 x_3$, namely, $(x_1 x_2) x_3$ and $x_1 (x_2 x_3)$.

We construct an infinite increasing sequence of submodules of A ,

$$(5.15) \quad S \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq A,$$

as follows. Define

$$S^1 = \text{span}_{\mathbb{K}}\{x_1 \cdots x_m \in A: m \geq 1, \text{ at least one } x_j \text{ in } S\}.$$

In other words, S^1 is the submodule of A generated by all the parenthesized monomials in A with at least one entry in S . Inductively, suppose that the submodules

$$S \subseteq S^1 \subseteq \cdots \subseteq S^n$$

have been defined for some $n \geq 1$. Then we set

$$S^{n+1} = \text{span}_{\mathbb{K}}\{x_1 \cdots x_m \in A: m \geq 1, \text{ at least one } x_j \text{ in } S^n \cup \alpha(S^n) \cup R(S^n)\}.$$

In other words, S^{n+1} is the submodule of A generated by all the parenthesized monomials in A with at least one entry in $S^n \cup \alpha(S^n) \cup R(S^n)$.

Now that we have an increasing sequence of submodules as in (5.15), we define

$$\langle S \rangle = \bigcup_{n \geq 1} S^n.$$

It is clear that $\langle S \rangle$ is a submodule of A that contains S . To see that $\langle S \rangle$ is an ideal in A , pick elements $x \in \langle S \rangle$ and $y \in A$. Then $x \in S^n$ for some finite n . Therefore, both $\alpha(x)$ and $R(x)$ lie in $S^{n+1} \subseteq \langle S \rangle$. Likewise, both xy and yx lie in $S^{n+1} \subseteq \langle S \rangle$. This shows that $\langle S \rangle$ is an ideal in A that contains S .

Suppose that I is another ideal in A that contains S . Since I is closed under multiplication with elements in A on either side, I also contains S^1 . Inductively, if I contains S^n , then it also contains $\alpha(S^n)$ and $R(S^n)$ because I is closed under both α and R . It follows that I contains S^{n+1} , again because I is closed under multiplication with elements in A on either side. By induction, I must contain $\langle S \rangle$ as well.

This proves the existence of the ideal $\langle S \rangle$. The uniqueness of $\langle S \rangle$ is obvious from its two properties in the statement of the Proposition. \square

Proof of Theorem 5.8. Pick an object $(A, \mu, \alpha, R) \in \mathbf{D}$. Let S be the subset of A consisting of the generating relations in a typical Rota-Baxter G -Hom-associative algebra, i.e., the elements

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mathbf{as}_A \circ \sigma(x, y, z)$$

and

$$R(x)R(y) - R(R(x)y + xR(y) + \lambda xy)$$

for all $x, y, z \in A$. Then we set

$$(5.16) \quad \mathcal{F}_2(A) = A/\langle S \rangle,$$

the quotient of A by the ideal $\langle S \rangle$ generated by S . The quotient $\mathcal{F}_2(A)$ is an object in \mathbf{D} with the induced operations from A . Moreover, it follows immediately from its construction that $\mathcal{F}_2(A)$ is an object in the subcategory \mathbf{HomRB}_λ . The construction of \mathcal{F}_2 is clearly functorial, so we have a functor $\mathcal{F}_2: \mathbf{D} \rightarrow \mathbf{HomRB}_\lambda$.

To show that \mathcal{F}_2 is the left adjoint of \mathcal{O}_2 , pick an object $B \in \mathbf{HomRB}_\lambda$. Let $f: A \rightarrow B \in \mathbf{D}$ be a morphism. We must show that there exists a unique morphism

$$\zeta: \mathcal{F}_2(A) \rightarrow B \in \mathbf{HomRB}_\lambda$$

such that

$$\zeta \circ p = f,$$

where $p: A \rightarrow \mathcal{F}_2(A)$ is the projection map. In other words, we must show that f factors through the quotient $A/\langle S \rangle$. It suffices to show that $\langle S \rangle$ is contained in the kernel of f . Since $\langle S \rangle$ is defined as the union $\cup_{n \geq 1} S^n$, it is enough to show that

$$f(S^n) = 0$$

for all $n \geq 1$.

It is clear that $f(S) = 0$, since $B \in \mathbf{HomRB}_\lambda$ and f commutes with μ , α , and R . This implies that $f(S^1) = 0$. Inductively, suppose that $f(S^n) = 0$. Then we have

$$f(\alpha(S^n)) = \alpha(f(S^n)) = 0$$

and, similarly, $f(R(S^n)) = 0$. This implies that $f(S^{n+1}) = 0$. This finishes the induction and shows that $f(S^n) = 0$ for all $n \geq 1$, as desired. \square

6. HOM-(TRI)DENDRIFORM ALGEBRAS

The purpose of this section is to study Hom-(tri)dendriform algebras. We discuss some construction results for Hom-(tri)dendriform algebras and observe that Hom-dendriform algebras give rise to Hom-associative and Hom-preLie algebras. Free Hom-(tri)dendriform algebras are also discussed.

6.1. Hom-dendriform algebras. Dendriform algebras were introduced by Loday in [54]. Dendriform algebras are algebras with two operations, which dichotomize the notion of associative algebra. We now generalize this structure by twisting the identities by a linear map.

Definition 6.2. A *Hom-dendriform algebra* is a quadruple

$$(A, \prec, \succ, \alpha)$$

consisting of a \mathbb{K} -module A and linear maps $\prec, \succ: A \otimes A \rightarrow A$ and $\alpha: A \rightarrow A$ that satisfy the identities

$$(6.1) \quad (x \prec y) \prec \alpha(z) = \alpha(x) \prec (y \prec z + y \succ z),$$

$$(6.2) \quad (x \succ y) \prec \alpha(z) = \alpha(x) \succ (y \prec z),$$

$$(6.3) \quad (x \prec y + x \succ y) \succ \alpha(z) = \alpha(x) \succ (y \succ z)$$

for x, y, z in A . A Hom-dendriform algebra is *multiplicative* if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ for $\mu = \prec$ and \succ .

Let $(A, \prec, \succ, \alpha)$ and $(A', \prec', \succ', \alpha')$ be two Hom-dendriform algebras. A *morphism* $f: A \rightarrow A'$ of Hom-dendriform algebras is a linear map such that

$$\prec' \circ (f \otimes f) = f \circ \prec, \quad \succ' \circ (f \otimes f) = f \circ \succ, \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

The category of Hom-dendriform algebras is denoted by **HomDend**.

A dendriform algebra is a Hom-dendriform algebra with $\alpha = \text{Id}$. In particular, dendriform algebras form a full-subcategory of **HomDend**.

The following results are the analogs of the construction results, Theorem 3.9 - Corollary 3.11, for Hom-dendriform algebras.

Theorem 6.3. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra and $\beta: A \rightarrow A$ be a morphism. Then*

$$A_\beta = (A, \prec_\beta = \beta \circ \prec, \succ_\beta = \beta \circ \succ, \beta\alpha)$$

is also a Hom-dendriform algebra, which is multiplicative if A is.

Moreover, suppose $(A', \prec', \succ', \alpha')$ is a Hom-dendriform algebra, $\beta': A' \rightarrow A'$ is a morphism, and $f: A \rightarrow A'$ is a morphism such that $f\beta = \beta'f$. Then

$$f: A_\beta \rightarrow A'_{\beta'}$$

is a morphism.

Proof. The Hom-dendriform axioms (6.1)-(6.3) for A_β are obtained from those of A by applying β^2 . Notice that this assertion does not use the commutativity of β with α . The assertions about multiplicativity and the morphism f are obvious. \square

The following two results are special cases of Theorem 6.3.

Corollary 6.4. *Let $(A, \prec, \succ, \alpha)$ be a multiplicative Hom-dendriform algebra. Then*

$$A_{\alpha^n} = (A, \alpha^n \circ \prec, \alpha^n \circ \succ, \alpha^{n+1})$$

is also a multiplicative Hom-dendriform algebra for each $n \geq 1$.

Proof. This is the $\beta = \alpha^n$ special case of Theorem 6.3. \square

Corollary 6.5. *Let (A, \prec, \succ) be a dendriform algebra and $\beta: A \rightarrow A$ be a morphism. Then*

$$A_\beta = (A, \beta \circ \prec, \beta \circ \succ, \beta)$$

is a multiplicative Hom-dendriform algebra.

Proof. This is the $\alpha = \text{Id}$ special case of Theorem 6.3. \square

We now show that Hom-dendriform algebra structures lead to Hom-associative algebra structures, hence to G -Hom-associative algebra structures. We provide also a connection to Hom-preLie algebras.

Theorem 6.6. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Define a linear map $\star: A \otimes A \rightarrow A$ by*

$$(6.4) \quad x \star y = x \prec y + x \succ y$$

for $x, y \in A$. Then

$$A_a = (A, \star, \alpha)$$

is a Hom-associative algebra, hence also a G -Hom-associative algebra for every subgroup G of \mathcal{S}_3 .

Proof. To check the Hom-associative identity (2.5), observe that for $x, y, z \in A$, the sum of the left-hand sides of the Hom-dendriform axioms (6.1) - (6.3) is

$$(x \star y) \star \alpha(z).$$

Likewise, the sum of the right-hand sides of the axioms (6.1) - (6.3) is

$$\alpha(x) \star (y \star z).$$

Therefore, the sum of the three Hom-dendriform axioms says

$$(x \star y) \star \alpha(z) = \alpha(x) \star (y \star z),$$

which is the Hom-associative identity for A_a . \square

The following result says that a Hom-dendriform algebra gives rise to a left Hom-preLie algebra via a mixed commutator.

Theorem 6.7. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Define the linear map $\triangleright: A \otimes A \rightarrow A$ by*

$$x \triangleright y = x \succ y - y \prec x$$

for $x, y \in A$. Then

$$A_l = (A, \triangleright, \alpha)$$

is a left Hom-preLie algebra.

Proof. To check the left Hom-preLie identity (2.6), observe that for $x, y \in A$ we have:

$$\begin{aligned} (6.5) \quad (x \triangleright y) \triangleright \alpha(z) &= (x \succ y - y \prec x) \triangleright \alpha(z) \\ &= (x \succ y) \triangleright \alpha(z) - (y \prec x) \triangleright \alpha(z) \\ &\quad - \alpha(z) \prec (x \succ y) + \alpha(z) \prec (y \prec x) \end{aligned}$$

and

$$\begin{aligned} (6.6) \quad \alpha(x) \triangleright (y \triangleright z) &= \alpha(x) \triangleright (y \succ z - z \prec y) \\ &= \alpha(x) \triangleright (y \succ z) - \alpha(x) \triangleright (z \prec y) \\ &\quad - (y \succ z) \prec \alpha(x) + (z \prec y) \prec \alpha(x) \\ &= (x \prec y) \triangleright \alpha(z) + (x \succ y) \triangleright \alpha(z) - \alpha(x) \triangleright (z \prec y) \\ &\quad - (y \succ z) \prec \alpha(x) + \alpha(z) \prec (y \prec x) + \alpha(z) \prec (y \succ x). \end{aligned}$$

In the last equality above, we used the Hom-dendriform axioms (6.1) and (6.3). Using (6.5), (6.6), and the remaining Hom-dendriform axiom (6.2), the Hom-associator of A_l becomes

$$\begin{aligned} \mathfrak{as}_{A_l}(x, y, z) &= (x \triangleright y) \triangleright \alpha(z) - \alpha(x) \triangleright (y \triangleright z) \\ &= -(x \prec y + y \prec x) \triangleright \alpha(z) \\ &\quad - \alpha(z) \prec (x \succ y + y \succ x) \\ &\quad + (x \succ z) \prec \alpha(y) + (y \succ z) \prec \alpha(x). \end{aligned}$$

The last expression is symmetric in the variables x and y , so A_l is a left Hom-preLie algebra. \square

The next result is the right Hom-preLie version of the previous result.

Theorem 6.8. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Define the linear map $\triangleleft: A \otimes A \rightarrow A$ by*

$$x \triangleleft y = x \prec y - y \succ x$$

for $x, y \in A$. Then

$$A_r = (A, \triangleleft, \alpha)$$

is a right Hom-preLie algebra.

Proof. The proof is essentially the same as that of Theorem 6.7, so we will omit the details. \square

Remark 6.9. Recall that G -Hom-associative algebras are automatically Hom-Lie-admissible algebras (Proposition 2.9). In particular, given a Hom-dendriform algebra A , the Hom-associative algebra A_a (Theorem 6.6), the left Hom-preLie algebra A_l (Theorem 6.7), and the right Hom-preLie algebra A_r (Theorem 6.8) are all Hom-Lie-admissible algebras. In fact, direct computation shows that their commutator Hom-Lie algebras (Remark 2.6) are equal.

6.10. Free Hom-dendriform algebras. Next we discuss the free Hom-dendriform algebra associated to a Hom-module.

Theorem 6.11. *The forgetful functor*

$$\mathcal{O}: \mathbf{HomDend} \rightarrow \mathbf{HomMod}$$

given by

$$\mathcal{O}(A, \prec, \succ, \alpha) = (A, \alpha)$$

admits a left adjoint.

Sketch of proof. The proof is similar to that of Theorem 5.1, so we only give a sketch. Instead of the category \mathbf{D} in section 5.2, here we use the intermediate category \mathbf{E} whose objects are quadruples

$$(A, \mu_l, \mu_r, \alpha),$$

in which A is a \mathbb{K} -module, both μ_l and μ_r are bilinear maps on A , and α is a linear self-map on A . The forgetful functor \mathcal{O} factors as the composition of two forgetful functors,

$$\mathcal{O}_2: \mathbf{HomDend} \rightarrow \mathbf{E} \quad \text{and} \quad \mathcal{O}_1: \mathbf{E} \rightarrow \mathbf{HomMod},$$

similar to (5.1).

The left adjoint

$$\mathcal{F}_1: \mathbf{HomMod} \rightarrow \mathbf{E}$$

of \mathcal{O}_1 is constructed with a suitable modification of the proof of Theorem 5.3. We define a *modified decorated n -tree* to mean a pair (ψ, f) , where ψ is an n -tree, and f is a function from the set of internal nodes (and not leaves) of ψ to the product

$$\mathbb{Z}_{\geq 0} \times \{l, r\}.$$

On these modified decorated trees, the operation α is defined as adding 1 to the integer component of the decoration of the lowest internal node. There are two grafting operations \vee_l and \vee_r , which assign the decorations $(0, l)$ and $(0, r)$, respectively, to the new lowest internal node. Similar to (5.7), each such modified decorated n -tree gives a way of multiplying n elements in an object in \mathbf{E} . At an internal node with decoration $(m, *)$, where $*$ = l or r , one uses the multiplication μ_* to multiply the two elements from the two branches above the node. Then one applies α^m to the resulting product. One defines \mathcal{F}_1 as in (5.8) using modified decorated trees and checks that it is the desired left adjoint of \mathcal{O}_1 .

For the left adjoint

$$\mathcal{F}_2: \mathbf{E} \rightarrow \mathbf{HomDend}$$

of \mathcal{O}_2 , one modifies the proof of Theorem 5.8. For an object $(A, \mu_l, \mu_r, \alpha) \in \mathbf{E}$, an *ideal* is a submodule I that is closed under α and such that

$$\mu_*(I, A) \subseteq I \quad \text{and} \quad \mu_*(A, I) \subseteq I$$

for $*$ \in $\{l, r\}$. There is an obvious analog of Proposition 5.10 for the existence and uniqueness of an ideal generated by a non-empty subset. One then defines \mathcal{F}_2 as in (5.16), where S here is the set of generating relations in a typical Hom-dendriform algebra (6.1) - (6.3). \square

6.12. Hom-tridendriform algebras. The notion of tridendriform algebra was introduced by Loday and Ronco in [55]. A tridendriform algebra is a vector space equipped with 3 binary operations \prec, \succ, \bullet satisfying seven relations. We extend this notion to the Hom situation as follows.

Definition 6.13. A *Hom-tridendriform algebra* is a quintuple

$$(A, \prec, \succ, \bullet, \alpha)$$

consisting of a \mathbb{K} -module A and linear maps $\prec, \succ, \bullet: A \otimes A \rightarrow A$ and $\alpha: A \rightarrow A$ that satisfy the following axioms for all $x, y, z \in A$:

$$(6.7) \quad (x \prec y) \prec \alpha(z) = \alpha(x) \prec (y \prec z + y \succ z + y \bullet z),$$

$$(6.8) \quad (x \succ y) \prec \alpha(z) = \alpha(x) \succ (y \prec z),$$

$$(6.9) \quad \alpha(x) \succ (y \succ z) = (x \prec y + x \succ y + x \bullet y) \succ \alpha(z),$$

$$(6.10) \quad (x \prec y) \bullet \alpha(z) = \alpha(x) \bullet (y \succ z),$$

$$(6.11) \quad (x \succ y) \bullet \alpha(z) = \alpha(x) \succ (y \bullet z),$$

$$(6.12) \quad (x \bullet y) \prec \alpha(z) = \alpha(x) \bullet (y \prec z),$$

$$(6.13) \quad (x \bullet y) \bullet \alpha(z) = \alpha(x) \bullet (y \bullet z).$$

A *morphism* of Hom-tridendriform algebras is a morphism of the underlying Hom-modules that is compatible with the three binary operations. The category of Hom-tridendriform algebras is denoted by **HomTridend**. A Hom-tridendriform algebra is *multiplicative* if the twisting map α is a morphism of Hom-tridendriform algebras. A *tridendriform algebra* is a Hom-tridendriform algebra with $\alpha = \text{Id}$.

Remark 6.14. Every Hom-tridendriform algebra gives a Hom-dendriform algebra by setting $x \bullet y = 0$ for any $x, y \in A$. Also, the binary operation \bullet satisfies the Hom-associativity identity (2.5) by the axiom (6.13).

Some of the results above for Hom-dendriform algebras have obvious analogs for Hom-tridendriform algebras, whose proofs are slight modifications of those given above. Therefore, we omit the proofs of the following results.

Theorem 6.15. Let $(A, \prec, \succ, \bullet, \alpha)$ be a Hom-tridendriform algebra and $\beta: A \rightarrow A$ be a morphism. Then

$$A_\beta = (A, \prec_\beta = \beta \circ \prec, \succ_\beta = \beta \circ \succ, \bullet_\beta = \beta \circ \bullet, \beta\alpha)$$

is also a Hom-tridendriform algebra, which is multiplicative if A is.

Moreover, suppose $(A', \prec', \succ', \bullet', \alpha')$ is a Hom-tridendriform algebra, $\beta': A' \rightarrow A'$ is a morphism, and $f: A \rightarrow A'$ is a morphism such that $f\beta = \beta'f$. Then

$$f: A_\beta \rightarrow A'_{\beta'}$$

is a morphism.

Corollary 6.16. Let $(A, \prec, \succ, \bullet, \alpha)$ be a multiplicative Hom-tridendriform algebra. Then

$$A_{\alpha^n} = (A, \alpha^n \circ \prec, \alpha^n \circ \succ, \alpha^n \circ \bullet, \alpha^{n+1})$$

is also a multiplicative Hom-tridendriform algebra for each $n \geq 1$.

Corollary 6.17. Let $(A, \prec, \succ, \bullet)$ be a tridendriform algebra and $\beta: A \rightarrow A$ be a morphism. Then

$$A_\beta = (A, \beta \circ \prec, \beta \circ \succ, \beta \circ \bullet, \beta)$$

is a multiplicative Hom-tridendriform algebra.

Theorem 6.18. Let $(A, \prec, \succ, \bullet, \alpha)$ be a Hom-tridendriform algebra. Define a linear map $*$: $A \otimes A \rightarrow A$ by

$$(6.14) \quad x * y = x \prec y + x \succ y + x \bullet y$$

for $x, y \in A$. Then

$$A_a = (A, *, \alpha)$$

is a Hom-associative algebra, hence also a G -Hom-associative algebra for every subgroup G of \mathcal{S}_3 .

Theorem 6.19. *The forgetful functor*

$$\mathcal{O}: \mathbf{HomTridend} \rightarrow \mathbf{HomMod}$$

given by

$$\mathcal{O}(A, \prec, \succ, \bullet, \alpha) = (A, \alpha)$$

admits a left adjoint.

7. ROTA-BAXTER HOM-ASSOCIATIVE AND HOM-DENDRIFORM ALGEBRAS

In this section, we discuss an adjunction between a full subcategory of the category of Rota-Baxter Hom-associative algebras and the category of Hom-(tri)dendriform algebras.

Denote by \mathbf{HomRBA}_λ the full subcategory of the category of Rota-Baxter Hom-associative algebras of weight λ , consisting of $(A, \mu, \alpha, R) \in \mathbf{HomRB}_\lambda$ in which the twisting map commutes with the Rota-Baxter operator, i.e., $\alpha R = R\alpha$.

Theorem 7.1. *There is an adjoint pair of functors*

$$(7.1) \quad U_{HD}: \mathbf{HomDend} \rightleftarrows \mathbf{HomRBA}_\lambda: HD,$$

in which the right adjoint is given by

$$HD(A, \mu, \alpha, R) = (A, \prec, \succ, \alpha) \in \mathbf{HomDend}$$

with

$$(7.2) \quad x \prec y = xR(y) + \lambda xy \quad \text{and} \quad x \succ y = R(x)y$$

for $x, y \in A$.

Recall that xy denotes $\mu(x, y)$.

Proof. First we need to show that $HD(A) = (A, \prec, \succ, \alpha)$ is a Hom-dendriform algebra for any $A = (A, \mu, \alpha, R) \in \mathbf{HomRBA}_\lambda$. In other words, we need to establish the three conditions (6.1)-(6.3) for $HD(A)$.

For the axiom (6.1), first note that for y and $z \in A$ we have

$$(7.3) \quad \begin{aligned} R(y \prec z + y \succ z) &= R(yR(z) + \lambda yz + R(y)z) \\ &= R(y)R(z) \end{aligned}$$

by the Rota-Baxter identity (3.1). Using (7.3) we compute as follows:

$$\begin{aligned} (x \prec y) \prec \alpha(z) &= (xR(y) + \lambda xy) R(\alpha(z)) + \lambda (xR(y) + \lambda xy) \alpha(z) \\ &= (xR(y))\alpha(R(z)) + \lambda (xy)\alpha(R(z)) + \lambda (xR(y))\alpha(z) + \lambda^2 (xy)\alpha(z) \\ &= \alpha(x)(R(y)R(z)) + \lambda \alpha(x)(yR(z)) + \lambda \alpha(x)(R(y)z) + \lambda^2 \alpha(x)(yz) \\ &= \alpha(x)R(y \prec z + y \succ z) + \lambda \alpha(x)(y \prec z + y \succ z) \\ &= \alpha(x) \prec (y \prec z + y \succ z). \end{aligned}$$

In the second equality above, we used $\alpha R = R\alpha$. In the third equality, we used the Hom-associativity axiom (2.5), which holds because $A \in \mathbf{HomRBA}_\lambda$. This proves (6.1) for $HD(A)$.

For (6.2) we compute as follows:

$$\begin{aligned}
 (x \succ y) \prec \alpha(z) &= (R(x)y)R(\alpha(z)) + \lambda(R(x)y)\alpha(z) \\
 &= (R(x)y)\alpha(R(z)) + \lambda\alpha(R(x))(yz) \\
 &= \alpha(R(x))(yR(z)) + R(\alpha(x))(\lambda yz) \\
 &= R(\alpha(x))(yR(z) + \lambda yz) \\
 &= \alpha(x) \succ (y \prec z).
 \end{aligned}$$

The Hom-associativity axiom (2.5) was used in the second and the third equalities above, and $\alpha R = R\alpha$ was used in the second to the fourth equalities. This proves (6.2) for $HD(A)$.

For (6.3) we compute as follows:

$$\begin{aligned}
 (x \prec y + x \succ y) \succ \alpha(z) &= R(x \prec y + x \succ y)\alpha(z) \\
 &= (R(x)R(y))\alpha(z) \\
 &= \alpha(R(x))(R(y)z) \\
 &= R(\alpha(x))(R(y)z) \\
 &= \alpha(x) \succ (y \succ z).
 \end{aligned}$$

We used (7.3) in the second equality, the Hom-associativity axiom (2.5) in the third equality, and $\alpha R = R\alpha$ in the fourth equality.

We have shown that $HD(A)$ is a Hom-dendriform algebra. The functoriality of HD is clear.

Now we show that HD admits a left adjoint

$$U_{HD}: \mathbf{HomDend} \rightarrow \mathbf{HomRBA}_\lambda.$$

Pick a Hom-dendriform algebra $(A, \prec, \succ, \alpha)$. Using Theorem 5.3, consider the free object

$$(\mathcal{F}_1(A), \mu, \alpha, R) \in \mathbf{D}$$

associated to the Hom-module (A, α) . Let S be the subset of $\mathcal{F}_1(A)$ consisting of:

- (1) $im(\mu \circ (\mu \otimes \alpha - \alpha \otimes \mu))$.
- (2) $im(\alpha \circ R - R \circ \alpha)$.
- (3) $im(\mu \circ R^{\otimes 2} - R \circ \mu \circ (R \otimes Id + Id \otimes R + \lambda Id^{\otimes 2}))$.
- (4) $x \prec y - (xR(y) + \lambda xy)$ for $x, y \in A$.
- (5) $x \succ y - R(x)y$ for $x, y \in A$.

Here we are identifying A as a submodule of $\mathcal{F}_1(A)$ via the inclusion ι (5.9). Let $\langle S \rangle$ be the ideal generated by S (as in Proposition 5.10).

From the definition of S , it is straightforward to see that the quotient

$$U_{HD}(A) = \mathcal{F}_1(A)/\langle S \rangle,$$

with the induced operations of μ , α , and R , is an object in \mathbf{HomRBA}_λ . The functoriality of U_{HD} is also clear. There is a natural map

$$i: A \xrightarrow{\iota} \mathcal{F}_1(A) \xrightarrow{pr} U_{HD}(A).$$

The proof that U_{HD} is the left adjoint of the functor HD is essentially identical to the last two paragraphs in the proof of Theorem 5.8, using the freeness of $\mathcal{F}_1(A)$ and the definitions (7.2). \square

The following result says that, under the condition that the twisting map commutes with the Rota-Baxter operator, a Rota-Baxter Hom-associative algebra can be given a new Hom-associative structure involving both R and λ .

Corollary 7.2. *Let (A, μ, α, R) be an object in \mathbf{HomRBA}_λ . Define a multiplication on A by*

$$x * y = xR(y) + R(x)y + \lambda xy$$

for $x, y \in A$. Then

$$A' = (A, *, \alpha)$$

is a Hom-associative algebra. Moreover, we have

$$R(x * y) = R(x)R(y) \quad \text{and} \quad \tilde{R}(x * y) = -\tilde{R}(x)\tilde{R}(y)$$

where $\tilde{R}(x) = -\lambda x - R(x)$.

Proof. That A' is a Hom-associative algebra is an immediate consequence of Theorems 6.6 and 7.1. The identity involving R is simply the Rota-Baxter identity (3.1). The identity involving \tilde{R} can be checked by the following computation:

$$\begin{aligned} \tilde{R}(x * y) &= -\lambda(x * y) - R(x * y) \\ &= -\lambda x R(y) - \lambda R(x)y - \lambda^2 xy - R(x)R(y) \\ &= -(-\lambda x - R(x))(-\lambda y - R(y)) \\ &= -\tilde{R}(x)\tilde{R}(y). \end{aligned}$$

□

The following result is the Hom-tridendriform analog of Theorem 7.1.

Theorem 7.3. *There is an adjoint pair of functors*

$$U_{HT} : \mathbf{HomTridend} \rightleftarrows \mathbf{HomRBA}_\lambda : HT,$$

in which U_{HT} is the left adjoint. For $(A, \mu, \alpha, R) \in \mathbf{HomRBA}_\lambda$, the three binary operations in the object

$$HT(A) = (A, \prec, \succ, \bullet, \alpha) \in \mathbf{HomTridend}$$

are defined as

$$x \prec y = xR(y), \quad x \succ y = R(x)y, \quad x \bullet y = \lambda xy$$

for $x, y \in A$.

Proof. The proof is very similar to that of Theorem 7.1, so we will omit the details. □

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